II. On nonlocality and incompatibility breaking channels

Definition 2.0. Let $\mathcal{H}$ be complex seperable Hilbert space and $\mathcal{L}(\mathcal{H})$ be the set of all bounded operator H. A postive operator $\varrho \in \mathcal{L}(\mathcal{H})$ of trace one is said to be a **state**, and the set of all states is denoted by $\mathcal{S}(\mathcal{H})$. A positive operator A that is bounded by the operator $\mathbb{I}$ is called an **effect**, the set of all effect is denoted by $\mathcal{E}(\mathcal{H})$, note that:

$$\mathcal{L}(\mathcal{H}) \subset \mathcal{S}(\mathcal{H}) \tag{1}$$

Definition 2.1 A class $\mathcal{A}$ of sets is said to be an **algebra** in a d-dimensional complex space if it has the following properties:

(a) $\emptyset \in \mathcal{A}$

(b) If $U \in \mathcal{A}$ and $V \in \mathcal{A}$, then $U - V \in \mathcal{A}$

(c) If $U \in \mathcal{A}$ and $V \in \mathcal{A}$, then $U \cup V \in \mathcal{A}$

(d) $C^d \in \mathcal{A}$

Furthurmore, an algebra is called an $\sigma-$algebra, then it has the following property:

$$\text{If } U_n \in \mathcal{A} \text{ for } n = 1, 2, ..., \text{then } \cup_{n=1}^{\infty} U_n \in \mathcal{A} \tag{2}$$

Note that since a quantum channel $\Lambda : \mathcal{L}(\mathcal{H}_{\mathcal{A}}) \to \mathcal{L}(\mathcal{H}_{\mathcal{B}})$ where $\mathcal{L}(\mathcal{H}_i)$ denotes the operator space of H_i, and since both the operator space of $H_{\mathcal{A}}$ and $H_{\mathcal{B}}$ belong to C*-algebra, the channel Λ can be seen as a mapping C*-algebra to iteself, and generally, only completely positive, trace-preserving (CPTP) map is considered.

Lemma 2.2. In a two qubits case, a unital channel $\mathcal{E}$ is said to be CHSH-nonlocality-breaking iff. the output state $\mathbb{I} \otimes \mathcal{E}(|\Phi^+\rangle \langle\Phi^+|)$ does not violate CHSH inequality. Such channels are useless for distributing correlations that can be violating CHSH inequality.

Proof. The correlation matrix of a general CHSH operator can be written as:

$$B = \mathbf{a} \otimes (\mathbf{b} + \mathbf{b}') + \mathbf{a} \otimes (\mathbf{b} - \mathbf{b}') \tag{3}$$

the CHSH operator can then be diagonalizes to:

$$\mathcal{B} = d_1\sigma_1 \otimes \sigma_1 + d_3\sigma_3 \otimes \sigma_3 \tag{4}$$

Proposition 2.2.1 The CHSH value $\mathrm{tr}(B\rho)$ is maximized by a MES.

Proof

And arbitrary pure state can be written as:

$$|\psi\rangle = R \otimes \mathbb{I} \left|\Phi^+\right\rangle \tag{5}$$

and hence the CHSH value can be bounded by Cauchy-Schwarz inequality:

$$\begin{aligned}
\langle\psi|\mathcal{B}|\psi\rangle &= d_1 \langle\psi|\sigma_1 \otimes \sigma_1|\psi\rangle + d_3 \langle\psi|\sigma_3 \otimes \sigma_3|\psi\rangle \\
&= d_1 \left\langle\Phi^+\middle|R^\dagger \otimes \mathbb{I}\sigma_1 \otimes \sigma_1 R \otimes \mathbb{I}\middle|\Phi^+\right\rangle + d_3 \left\langle\Phi^+\middle|R^\dagger \otimes \mathbb{I}\sigma_3 \otimes \sigma_3 R \otimes \mathbb{I}\middle|\Phi^+\right\rangle \\
&= d_1 \mathrm{tr}(R^\dagger \sigma_1 R \sigma_1) + d_3 \mathrm{tr}(R^\dagger \sigma_3 R \sigma_3) \\
&\leq (d_1 + d_3)\mathrm{tr}(R^\dagger R) \leq d_1 + d_3
\end{aligned} \tag{6}$$

Where the the bound is satisfied as $R = \mathbb{I}$, meaning that the CHSH value is maximized by a MES.

Then if the channel $\mathcal{E}$ is unital, then the state:

$$\mathrm{tr}(\mathcal{B}(\mathbb{I} \otimes \mathcal{E} \left|\Phi^+\right\rangle \left\langle\Phi^+\right|)) = \mathrm{tr}(\left|\Phi^+\right\rangle \left\langle\Phi^+\right| \mathbb{I} \otimes \mathcal{E}^\dagger(\mathcal{B})) \tag{7}$$

with $\mathcal{E}^\dagger$ being the dual map of $\mathcal{E}$, and since $\mathcal{E}$ is unital, $\mathrm{tr}(\mathcal{E}(\mathcal{B})) = 0$, and similarly by the proof of Proposition 2.2.1, it can be seen that the CHSH value is still bounded by MES. □

Definition 2.3 Joint measurability is usually seen as the equivalent of commutativity, while it is true if the observables are represented solely by self-adjoint operators, generally it's not true for

Definition 2.4 A set of observables $\mathcal{A} = \{A_1, A_2, ..., A_n\}$ is said to be **compatible** iff. $\exists$ an observable G $\ni$ $G(X_1, \Omega_{A_2}, ..., \Omega_{A_n}) = A_1$, $G(\Omega_{A_1}, X_2, ..., \Omega_{A_n}) = A_2, \ldots, G(\Omega_{A_1}, \Omega_{A_2}, ..., X_n) = A_n$